Completely Integrable Equation for the Quantum Correlation Function of Nonlinear Schrödinger Eqaution.

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Abstract

Correlation functions of exactly solvable models can be described by differential equations [1]. In this paper we show that for non free fermionic case differential equations should be replaced by integro-differential equations. We derive an integro-differential equation, which describes time and temperature dependent correlation function $\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T$ of penetrable Bose gas. The integro-differential equation turns out be the continuum generalization of classical nonlinear Schrödinger equation.

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1 Introduction

We consider exactly solvable models of statistical mechanics in one space and one time dimension. The Quantum Inverse Scattering Method and Algebraic Bethe Ansatz are effective methods for a description of the spectrum of these models. Our aim is the evaluation of correlation functions of exactly solvable models. Our approach is based on the determinant representation for correlation functions. It consists of a few steps: first the correlation function is represented as a determinant of a Fredholm integral operator, second the Fredholm integral operator is described by a classical completely integrable equation, third the classical completely integrable equation is solved by means of the Riemann–Hilbert problem. This permits us to evaluate the long distance and large time asymptotics of the correlation function. The method is described in [7]. The most interesting correlation functions are time dependent correlation functions. The determinant representation for time and temperature dependent correlation functions of quantum nonlinear Schrödinger equation was obtained in [8]. In this paper we describe the correlation function by means of completely integrable integro-differential equation. In the forthcoming publication we shall formulate the Riemann–Hilbert problem for this equation and evaluate long distance asymptotic.

The quantum nonlinear Schrödinger equation can be described in terms of canonical Bose fields $\psi(x,t), \psi^\dagger(x,t)$ ($x \in \mathbf{R}$) obeying

$$[\psi(x,t), \psi^\dagger(y,t)] = \delta(x-y). \tag{1.1}$$

The Hamiltonian and momentum of the model are

$$H = \int dx \left(\partial_x \psi^\dagger(x) \partial_x \psi(x) + c \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) - h \psi^\dagger(x) \psi(x) \right), \tag{1.2}$$

$$P = -i \int dx \psi^\dagger(x) \partial_x \psi(x). \tag{1.3}$$

Here $0 < c \leq \infty$ is the coupling constant and $h > 0$ is the chemical potential. The spectrum of the model was first described by E. H. Lieb and W. Liniger [13], [14]. The Lax representation for the corresponding classical equation of motion

$$i \frac{\partial}{\partial t} \psi = [\psi, H] = -\frac{\partial^2}{\partial x^2} \psi + 2c \psi^\dagger \psi \psi - h \psi, \tag{1.4}$$

was found by V. E. Zakharov and A. B. Shabat [17]. The Quantum Inverse Scattering Method for the model was formulated by L. D. Faddeev and E. K. Sklyanin [2]. The quantum nonlinear Schrödinger equation is equivalent to the Bose gas with delta-function interaction. In the sector

with N particles the Hamiltonian of Bose gas is given by

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + 2c \sum_{1\le j<k\le N} \delta(z_k - z_j) - Nh. \tag{1.5}$$

In this paper we shall consider the thermodynamic of the model. The partition function and the free energy of the model are defined by

$$Z = \operatorname{tr} e^{-\frac{H}{T}} = e^{-\frac{F}{T}}. \tag{1.6}$$

The free energy F can be expressed in terms of Yang-Yang equation [16]

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi}\int_{-\infty}^{\infty} \frac{2c}{c^2+(\lambda-\mu)^2} \ln\left(1+e^{-\frac{\varepsilon(\mu)}{T}}\right) d\mu, \tag{1.7}$$

$$F = -\frac{T}{2\pi}\int_{-\infty}^{\infty} \ln\left(1+e^{-\frac{\varepsilon(\mu)}{T}}\right) d\mu. \tag{1.8}$$

The correlation function, which we shall study in this paper, is defined by

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T = \frac{\operatorname{tr}\left(e^{-\frac{H}{T}}\psi(0,0)\psi^\dagger(x,t)\right)}{\operatorname{tr} e^{-\frac{H}{T}}}. \tag{1.9}$$

In the previous paper [8] we obtained the determinant representation for this correlation function. In this paper we shall derive completely integrable equations, starting from the determinant representation. The plan of this paper is the following. In Section 2 we shall remind the reader the determinant representation and definition of dual fields. In Section 3 we introduce new Hilbert space and rewrite the kernel of the integral operator in the canonical form. In Section 4 we define the resolvent of the integral operator. Section 5 is devoted to the construction of the Lax representation. In Section 6 we find the logarithmic derivatives of the Fredholm determinant and obtain the completely integrable equation describing the correlation function. In Section 7 we summarize the main results. In Appendix A we find some identities for potentials. We give the treatment of the quantum nonlinear Schrödinger equation as a continuum generalization of the classical equation in Appendix B. Appendix C is devoted to the free fermion limit.

2 Determinant representation for the correlation function

Our starting point is the determinant representation for the temperature correlation function of local fields obtained in [8]

$$
\begin{aligned}
\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T = e^{-iht}(0|\left(G(x,t)+\frac{\partial}{\partial\alpha}\right)\\
\times\frac{\det\left(\tilde I+\widetilde V-\frac{\alpha}{2\pi}\widetilde Y\right)}{\det\left(\tilde I-\frac{1}{2\pi}\widetilde{K_T}\right)}|0)\Bigg|_{\alpha=0}.
\end{aligned}
\tag{2.1}
$$

Let us explain our notations.

We begin by the numerator in the r.h.s. of (2.1), is the Fredholm determinant of the integral operator $\tilde I+\widetilde V-\frac{\alpha}{2\pi}\widetilde Y$ (here $\tilde I$ is identical operator: $I(\lambda,\mu)=\delta(\lambda-\mu)$). This operator acts on the real axis. The left and right actions on some trial function f are given by

$$
\begin{aligned}
\left(\tilde I+\widetilde V-\frac{\alpha}{2\pi}\widetilde Y\right)\circ f(\mu) = f(\lambda)+\int_{-\infty}^{\infty}\left(V(\lambda,\mu)-\frac{\alpha}{2\pi}Y(\lambda,\mu)\right)f(\mu)\,d\mu,\\
f(\lambda)\circ\left(\tilde I+\widetilde V-\frac{\alpha}{2\pi}\widetilde Y\right) = f(\mu)+\int_{-\infty}^{\infty}f(\lambda)\left(V(\lambda,\mu)-\frac{\alpha}{2\pi}Y(\lambda,\mu)\right)d\lambda.
\end{aligned}
\tag{2.2}
$$

Here and hereafter we denote by the symbol "$\circ$" the action of integral operators on functions.

The kernels of operators $\widetilde V$ and $\widetilde Y$ can be written in terms of auxiliary quantum operators — dual fields, acting in an auxiliary Fock space. One can find the detailed definition and properties of dual fields in Section 5 and Appendix C of [8]. Here we repeat them in brief.

Consider an auxiliary Fock space having vacuum vector $|0)$ and dual vector $(0|$. Three dual fields $\psi(\lambda)$, $\phi_{D_1}(\lambda)$ and $\phi_{A_2}(\lambda)$ acting in this space are defined as

$$
\begin{aligned}
\phi_{A_2}(\lambda) &= q_{A_2}(\lambda)+p_{D_2}(\lambda),\\
\phi_{D_1}(\lambda) &= q_{D_1}(\lambda)+p_{A_1}(\lambda),\\
\psi(\lambda) &= q_\psi(\lambda)+p_\psi(\lambda).
\end{aligned}
\tag{2.3}
$$

Here $p(\lambda)$ are annihilation parts of dual fields: $p(\lambda)|0)=0$; $q(\lambda)$ are creation parts of dual fields: $(0|q(\lambda)=0$. Thus, any dual field is the sum of annihilation and creation parts. Nonzero commutation relations are (see [8])

$$
\begin{aligned}
[p_{A_1}(\lambda),q_\psi(\mu)] &= \ln h(\mu,\lambda),\\
[p_{D_2}(\lambda),q_\psi(\mu)] &= \ln h(\lambda,\mu),\\
[p_\psi(\lambda),q_{A_2}(\mu)] &= \ln h(\mu,\lambda), \qquad \text{where} \qquad h(\lambda,\mu)=\frac{ic}{\lambda-\mu+ic}\\
[p_\psi(\lambda),q_{D_1}(\mu)] &= \ln h(\lambda,\mu),\\
[p_\psi(\lambda),q_\psi(\mu)] &= \ln[h(\lambda,\mu)h(\mu,\lambda)].
\end{aligned}
\tag{2.4}
$$

Recall that c is the coupling constant in (1.2). It follows immediately from (2.4) that the dual fields belong to an Abelian sub-algebra

$$[\psi(\lambda),\psi(\mu)]=[\psi(\lambda),\phi_a(\mu)]=[\phi_b(\lambda),\phi_a(\mu)]=0, \tag{2.5}$$

where $a,b=A_2,D_1$. This property, in fact, permits us to treat the dual fields as some c-number functions.

Let us define function $Z(\lambda,\mu)$:

$$Z(\lambda,\mu)=\frac{e^{-\phi_{D_1}(\lambda)}}{h(\mu,\lambda)}+\frac{e^{-\phi_{A_2}(\lambda)}}{h(\lambda,\mu)}. \tag{2.6}$$

The kernel $\widetilde{V}$ is equal to

$$V(\lambda,\mu)=\frac{e^{\frac{1}{2}(\phi_{D_1}(\lambda)+\phi_{A_2}(\lambda))}e^{\frac{1}{2}(\phi_{D_1}(\mu)+\phi_{A_2}(\mu))}\sqrt{\theta(\lambda)}\sqrt{\theta(\mu)}}{4\pi^2(\lambda-\mu)}$$

$$\times\int\limits_{-\infty}^{\infty}\frac{du}{Z(u,u)}\left(\frac{e^{-\phi_{D_1}(u)}}{u-\lambda-i0}+\frac{e^{-\phi_{A_2}(u)}}{u-\lambda+i0}-\frac{e^{-\phi_{D_1}(u)}}{u-\mu-i0}-\frac{e^{-\phi_{A_2}(u)}}{u-\mu+i0}\right)$$

$$\times e^{\psi(u)+\tau(u)}e^{-\frac{1}{2}(\psi(\lambda)+\tau(\lambda)+\psi(\mu)+\tau(\mu))}Z(u,\lambda)Z(u,\mu). \tag{2.7}$$

Integral operator $\widetilde{Y}$ is one-dimensional projector

$$Y(\lambda,\mu)=P(\lambda)P(\mu), \tag{2.8}$$

where

$$P(\mu)=\frac{e^{\frac{1}{2}(\phi_{D_1}(\mu)+\phi_{A_2}(\mu))}\sqrt{\theta(\mu)}}{2\pi}\int\limits_{-\infty}^{\infty}\frac{du}{Z(u,u)}\left(\frac{e^{-\phi_{D_1}(u)}}{u-\mu-i0}+\frac{e^{-\phi_{A_2}(u)}}{u-\mu+i0}\right)$$

$$\times e^{\psi(u)+\tau(u)}e^{-\frac{1}{2}(\psi(\mu)+\tau(\mu))}Z(u,\mu). \tag{2.9}$$

Here functions $\theta(\lambda)$ and $\tau(\lambda)$ are equal to

$$\theta(\lambda) = \left(1+\exp\left[\frac{\varepsilon(\lambda)}{T}\right]\right)^{-1}, \tag{2.10}$$

$$\tau(\lambda) = it\lambda^2-ix\lambda. \tag{2.11}$$

The Fermi weight $\theta(\lambda)$ defines the dependence of the correlation function on temperature T. The energy of one-particle excitation $\varepsilon(\lambda)$ is given in (1.7). The function $\tau(\lambda)$ depends also on the

distance x and the time t. All other functions entering expressions for $V(\lambda,\mu)$ and $P(\mu)$ do not depend on x and t.

We would like to draw reader's attention to the fact that formulæ (2.7) and (2.9) slightly differ from formulæ (6.25) and (6.26) of [8]. It is explained in the Appendixes C and D of [8] how one can reduce formulæ (6.25) and (6.26) to formulæ (2.7) and (2.9). Thus, the integral operator $\tilde{I}+\widetilde{V}-\frac{\alpha}{2\pi}\widetilde{Y}$ is explained.

The operator $\tilde{I}-\frac{1}{2\pi}\widetilde{K_T}$ also acts on the whole real axis. Its kernel is given by

$$K_T(\lambda,\mu) = \left(\frac{2c}{c^2+(\lambda-\mu)^2}\right)\sqrt{\theta(\lambda)}\sqrt{\theta(\mu)}. \tag{2.12}$$

Finally the function $G(x,t)$ in the r.h.s. of (2.1) is equal to

$$G(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{\psi(v)+\tau(v)}\,dv. \tag{2.13}$$

Thus, we have described the r.h.s. of (2.1). The temperature correlation function of local fields is proportional to the vacuum expectation in the auxiliary Fock space of the Fredholm determinant of the integral operator. The auxiliary quantum operators—dual fields—enters the kernels $\widetilde{V}$ and $\widetilde{Y}$. Due to the property (2.5) the Fredholm determinant is well defined. Our aim now is a description of the correlation function in terms of solutions of classical completely integrable equations.

3 Vectors and operators of new Hilbert space

Introducing functions $E_\pm$:

$$E_+(\lambda|u) = \frac{1}{2\pi}\frac{Z(u,\lambda)}{Z(u,u)}\left(\frac{e^{-\phi_{A_2}(u)}}{u-\lambda+i0}+\frac{e^{-\phi_{D_1}(u)}}{u-\lambda-i0}\right)\sqrt{\theta(\lambda)}$$

$$\times e^{\psi(u)+\tau(u)+\frac{1}{2}(\phi_{D_1}(\lambda)+\phi_{A_2}(\lambda)-\psi(\lambda)-\tau(\lambda))}, \tag{3.1}$$

$$E_-(\lambda|u) = \frac{1}{2\pi}Z(u,\lambda)e^{\frac{1}{2}(\phi_{D_1}(\lambda)+\phi_{A_2}(\lambda)-\psi(\lambda)-\tau(\lambda))}\sqrt{\theta(\lambda)}. \tag{3.2}$$

The functions $E_\pm$ depend also on the distance x, the time t, the temperature T and the chemical potential h, but this dependence as a rule is suppressed in the notation. One can rewrite expressions (2.7) and (2.9) for $V(\lambda,\mu)$ and $P(\mu)$ in terms of these functions

$$V(\lambda,\mu) = \frac{1}{\lambda-\mu}\int\limits_{-\infty}^{\infty} du(E_+(\lambda|u)E_-(\mu|u)-E_-(\lambda|u)E_+(\mu|u)), \tag{3.3}$$

$$Y(\lambda,\mu) = P(\lambda)P(\mu) = \int\limits_{-\infty}^{\infty} dudvE_+(\lambda|u)E_+(\mu|v). \tag{3.4}$$

Using obvious equalities

$$\partial_x e^{\tau(\lambda)} = -i\lambda e^{\tau(\lambda)}; \qquad \partial_t e^{\tau(\lambda)} = i\lambda^2 e^{\tau(\lambda)},$$

we arrive at

$$\begin{aligned} \partial_x E_+(\lambda|u) &= -\tfrac{i\lambda}{2}E_+(\lambda|u) - i\int\limits_{-\infty}^{\infty} dvg(u,v)E_-(\lambda|v), \\ \partial_x E_-(\lambda|u) &= \tfrac{i\lambda}{2}E_-(\lambda|u), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \partial_t E_+(\lambda|u) &= \tfrac{i\lambda^2}{2}E_+(\lambda|u) + \int\limits_{-\infty}^{\infty} dv(i\lambda g(u,v) - \partial_x g(u,v))E_-(\lambda|v), \\ \partial_t E_-(\lambda|u) &= -\tfrac{i\lambda^2}{2}E_-(\lambda|u). \end{aligned} \tag{3.6}$$

Here

$$g(u,v) = \delta(u-v)e^{\psi(v)+\tau(v)}. \tag{3.7}$$

The relations (3.5) and (3.6) are important for a description of the correlation function in terms of solutions of completely integrable equations. In order to do this it is necessary to investigate properties of the integral operator $\tilde{I}+\widetilde{V}$. Later on we shall show how one can take into consideration the contribution of the projector $\widetilde{Y}$.

In order to derive differential equations for the correlation function it is convenient to treat functions $E_\pm$ as components of vectors of some Hilbert space $\mathcal{H}$, for example, rigged $L_2(-\infty,\infty)\otimes R_2$ [3]. Let us introduce the bra-vector $\langle E^L(\lambda)|$ and the ket-vector $|E^R(\lambda)\rangle$ belong to the Hilbert space $\mathcal{H}$. Both of these vectors have two discrete components (corresponding to the space R_2), which we shall denote with indices 1 and 2. In turn any discrete component has continuous "index" (corresponding to the rigged $L_2(-\infty,\infty)$ space) which we shall denote as "u" (or "v", "w" etc.). So

$$\langle E^L(\lambda)| = \Big(E_1^L(\lambda|u), E_2^L(\lambda|u)\Big); \qquad |E^R(\mu)\rangle = \begin{pmatrix} E_1^R(\mu|u) \\ \\ E_2^R(\mu|u) \end{pmatrix}.$$

The definition of the scalar product is standard :

$$\langle E^L(\lambda)|E^R(\mu)\rangle = \int\limits_{-\infty}^{\infty} du\left(E_1^L(\lambda|u)E_1^R(\mu|u) + E_2^L(\lambda|u)E_2^R(\mu|u)\right). \tag{3.8}$$

On the contrary the products of type $|E^R(\mu)\rangle\langle E^L(\lambda)|$ as usual are operators in Hilbert space $\mathcal{H}$. We shall consider such operators below.

Let us identify

$$
\begin{aligned}
E_1^R(\mu|u) &= E_+(\mu|u), & E_1^L(\lambda|u) &= -E_-(\lambda|u), \\
E_2^R(\mu|u) &= E_-(\mu|u), & E_2^L(\lambda|u) &= E_+(\lambda|u).
\end{aligned}
\tag{3.9}
$$

Then one can rewrite the kernel of the integral operator V as

$$
V(\lambda,\mu) = \frac{\langle E^L(\lambda)|E^R(\mu)\rangle}{\lambda-\mu}. \tag{3.10}
$$

Due to (3.9) we have

$$
\langle E^L(\lambda)|E^R(\lambda)\rangle = 0 \tag{3.11}
$$

and hence the kernel $V(\lambda,\mu)$ is not singular in the point $\lambda=\mu$.

The representation (3.10) is the canonical form of the kernels of the completely integrable integral operators. In all examples related to correlation functions, the kernels of integral operators can be presented in the form (3.10). The different realizations of space $\mathcal{H}$ correspond to the concrete correlation functions. For example, in the free fermion situation (the coupling constant c goes to infinity) $\mathcal{H} = R_n$, where n is the number of fields [4], [5], [6], [9], [15]. For equal-time correlation functions of penetrable bosons the representations of type (3.10) were constructed with $\mathcal{H} = L_2(0,\infty)\otimes R_{2n}$ in [6], [10], [11] (see also Section XIV of [7]). In the present paper we shall follow the method developed in the papers enumerated above.

Operators acting in the space $\mathcal{H}$ are defined in the standard way. They have discrete and continuous indices: $\widehat{A} = A_{jk}(u,v), \quad j,k = 1,2; \quad -\infty < u,v < \infty$. We shall denote these operators with the sign "hat" in order to distinguish them from integral operators which we have denoted with the sign "tilde". Action on vectors is given by

$$
\begin{aligned}
\widehat{A}|E^R(\mu)\rangle &= \sum_{k=1}^{2}\int\limits_{-\infty}^{\infty} A_{jk}(u,v)E_k^R(\mu|v)\,dv, \\
\langle E^L(\lambda)|\widehat{A} &= \sum_{j=1}^{2}\int\limits_{-\infty}^{\infty} E_j^L(\lambda|u)A_{jk}(u,v)\,du.
\end{aligned}
\tag{3.12}
$$

On the contrary the integral operators of type $\tilde{I}+\widetilde{V}$ appear to be scalars relatively to the space $\mathcal{H}$,

for example

$$\left(\tilde{I}+\widetilde{V}\right)\circ|E^R(\mu)\rangle=\left(\begin{array}{c} E_1^R(\lambda|u)+\int\limits_{-\infty}^{\infty}V(\lambda,\mu)E_1^R(\mu|u)\,d\mu \\ \\ E_2^R(\lambda|u)+\int\limits_{-\infty}^{\infty}V(\lambda,\mu)E_2^R(\mu|u)\,d\mu \end{array}\right).$$

The product of operators of type $\hat{A}$ is

$$\widehat{A}^{(1)}\widehat{A}^{(2)}=\sum_{l=1}^{2}\int\limits_{-\infty}^{\infty}A_{jl}^{(1)}(u,w)A_{lk}^{(2)}(w,v)\,dw.$$

The trace of operator is defined as usual

$$\mathrm{tr}\,\widehat{A}=\int\limits_{-\infty}^{\infty}du\Big(A_{11}(u,u)+A_{22}(u,u)\Big).$$

In particular for operators of type $|\ldots\rangle\langle\ldots|$ we have

$$\mathrm{tr}\,|\ldots\rangle\langle\ldots|=\langle\ldots|\ldots\rangle.$$

Using the operator notations, one can rewrite the relations (3.5), (3.6) in the form of linear partial differential equations :

$$\partial_x|E^R(\lambda)\rangle=\widehat{L}(\lambda)|E^R(\lambda)\rangle,\qquad \partial_x\langle E^L(\lambda)|=-\langle E^L(\lambda)|\widehat{L}(\lambda), \tag{3.13}$$

$$\partial_t|E^R(\lambda)\rangle=\hat{M}(\lambda)|E^R(\lambda)\rangle,\qquad \partial_t\langle E^L(\lambda)|=-\langle E^L(\lambda)|\hat{M}(\lambda), \tag{3.14}$$

where

$$\widehat{L}(\lambda)=\lambda\hat{\sigma}+[\hat{g},\hat{\sigma}], \tag{3.15}$$

$$\hat{M}(\lambda)=-\lambda\widehat{L}(\lambda)+\partial_x\hat{g}, \tag{3.16}$$

and the operators $\hat{\sigma}$ and $\hat{g}$ are equal to

$$\hat{\sigma}=-\frac{i}{2}\sigma_3\delta(u-v)=-\frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\delta(u-v), \tag{3.17}$$

$$\hat{g}=-\sigma_+g(u,v)=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}g(u,v). \tag{3.18}$$

Later we shall use the relations (3.13) and (3.14) in order to construct the nontrivial Lax representation.

4 Vectors $\langle F^L(\lambda)|, \quad |F^R(\lambda)\rangle$ and resolvent

Let us introduce the vectors $\langle F^L(\lambda)|$ and $|F^R(\lambda)\rangle$ belong to the same space $\mathcal{H}$

$$\langle F^L(\lambda)| = \Big(F_1^L(\lambda|u), F_2^L(\lambda|u)\Big); \qquad |F^R(\mu)\rangle = \begin{pmatrix} F_1^R(\mu|u) \\ \\ F_2^R(\mu|u) \end{pmatrix}, \tag{4.1}$$

defining them as solutions of the integral equations

$$\left(\tilde{I} + \widetilde{V}\right) \circ \langle F^L(\mu)| = \langle E^L(\lambda)|, \tag{4.2}$$

$$|F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) = |E^R(\mu)\rangle. \tag{4.3}$$

More preciously these formulæ mean

$$F_j^L(\lambda) + \int\limits_{-\infty}^{\infty} V(\lambda,\mu) F_j^L(\mu)\, d\mu = E_j^L(\lambda),$$

$$F_j^R(\mu) + \int\limits_{-\infty}^{\infty} F_j^R(\lambda) V(\lambda,\mu)\, d\lambda = E_j^R(\mu).$$

Define the resolvent of the operator $\tilde{I} - \widetilde{V}$ as

$$\left(\tilde{I} + \widetilde{V}\right) \circ \left(\tilde{I} - \widetilde{R}\right) = \tilde{I}. \tag{4.4}$$

Obviously

$$\left(\tilde{I} - \widetilde{R}\right) \circ \langle E^L(\mu)| = \langle F^L(\lambda)|, \tag{4.5}$$

$$|E^R(\lambda)\rangle \circ \left(\tilde{I} - \widetilde{R}\right) = |F^R(\mu)\rangle. \tag{4.6}$$

Let us find the kernel of the resolvent. One can rewrite (4.4) as follows

$$V(\lambda,\mu) - \int\limits_{-\infty}^{\infty} V(\lambda,\nu) R(\nu,\mu)\, d\nu = R(\lambda,\mu).$$

Multiplying both sides of the last equality by $\lambda - \mu$ we get

$$(\lambda-\mu)V(\lambda,\mu) - \int\limits_{-\infty}^{\infty} V(\lambda,\nu)(\lambda - \nu + \nu - \mu) R(\nu,\mu)\, d\nu = (\lambda-\mu) R(\lambda,\mu),$$

or

$$\langle E^L(\lambda)|E^R(\mu)\rangle - \int\limits_{-\infty}^{\infty} \langle E^L(\lambda)|E^R(\nu)\rangle R(\nu,\mu)\, d\nu = \left(\tilde{I} + \widetilde{V}\right) \circ (\nu - \mu) R(\nu,\mu),$$

or due to (4.6)

$$\langle E^L(\lambda)|F^R(\mu)\rangle = \left(\tilde{I} + \widetilde{V}\right) \circ (\nu - \mu) R(\nu,\mu).$$

Making $\tilde{I} - \widetilde{R}$ act on this equation from the left, we get

$$\langle F^L(\lambda)|F^R(\mu)\rangle = (\lambda - \mu) R(\lambda,\mu).$$

Therefore we have come to the following expression for the resolvent kernel:

$$R(\lambda,\mu) = \frac{\langle F^L(\lambda)|F^R(\mu)\rangle}{\lambda - \mu}. \tag{4.7}$$

It is worth mentioning that this method of calculation of resolvent is a direct generalization of the method described in the Section XIV.1 of [7].

In the next Section we shall need the operator $\widehat{B}$ (potential), defined as

$$\widehat{B} = \int\limits_{-\infty}^{\infty} |F^R(\lambda)\rangle\langle E^L(\lambda)|\, d\lambda. \tag{4.8}$$

Obviously

$$\widehat{B} = |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) \circ \langle F^L(\mu)| = \int\limits_{-\infty}^{\infty} |E^R(\mu)\rangle\langle F^L(\mu)|\, d\mu. \tag{4.9}$$

The components of this operator are

$$B_{jk}(u,v) = \int\limits_{-\infty}^{\infty} F_j^R(\lambda|u) E_k^L(\lambda|v)\, d\lambda, \tag{4.10}$$

so

$$\widehat{B} = \begin{pmatrix} B_{11}(u,v) & B_{12}(u,v) \\ B_{21}(u,v) & B_{22}(u,v) \end{pmatrix}. \tag{4.11}$$

The operator $\widehat{C}$ defined in the similar way is also useful

$$\widehat{C} = \int\limits_{-\infty}^{\infty} \lambda |F^R(\lambda)\rangle\langle E^L(\lambda)|\, d\lambda. \tag{4.12}$$

The components of the operator $\widehat{C}$ are

$$C_{jk}(u,v) = \int\limits_{-\infty}^{\infty} \lambda F_j^R(\lambda|u) E_k^L(\lambda|v)\, d\lambda, \tag{4.13}$$

so

$$\widehat{C} = \begin{pmatrix} C_{11}(u,v) & C_{12}(u,v) \\ C_{21}(u,v) & C_{22}(u,v) \end{pmatrix}. \tag{4.14}$$

5 The Lax representation

In this Section we construct the Lax representation having nontrivial compatibility condition. Namely, we establish the following relations

$$\partial_x |F^R(\lambda)\rangle = \widehat{\mathcal{L}}(\lambda)|F^R(\lambda)\rangle, \tag{5.1}$$

$$\partial_t |F^R(\lambda)\rangle = \hat{\mathcal{M}}(\lambda)|F^R(\lambda)\rangle, \tag{5.2}$$

which are analogous to the relations (3.13) and (3.14). We shall prove that one can obtain the operators $\widehat{\mathcal{L}}(\lambda)$ and $\hat{\mathcal{M}}(\lambda)$ using formulæ (3.15) and (3.16) with replacement $\hat{g}$ by $\hat{g} + \widehat{B}$.

Let us calculate the derivative of $V(\lambda, \mu)$ with respect to x using formulæ (3.13)

$$\begin{aligned} \partial_x V(\lambda,\mu) &= -\frac{\langle E^L(\lambda)|(\widehat{L}(\lambda) - \widehat{L}(\mu))|E^R(\mu)\rangle}{\lambda - \mu} \\ &= -\langle E^L(\lambda)|\hat{\sigma}|E^R(\mu)\rangle. \end{aligned} \tag{5.3}$$

Thus, from (4.3) we get

$$\partial_x |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) - \int_{-\infty}^{\infty} |F^R(\lambda)\rangle\langle E^L(\lambda)|\hat{\sigma}|E^R(\mu)\rangle \, d\lambda = \widehat{L}(\mu)|E^R(\mu)\rangle, \tag{5.4}$$

or using the definition of $\widehat{B}$

$$\partial_x |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) - \widehat{B}\cdot\hat{\sigma}|E^R(\mu)\rangle = \widehat{L}(\mu)|E^R(\mu)\rangle.$$

Making $\tilde{I} - \widetilde{R}$ on this equality the right, we get

$$\partial_x |F^R(\lambda)\rangle - \widehat{B}\cdot\hat{\sigma}|F^R(\lambda)\rangle = (\widehat{L}(\mu) - \widehat{L}(\lambda))|E^R(\mu)\rangle \circ \left(\tilde{I} - \widetilde{R}\right) + \widehat{L}(\lambda)|F^R(\lambda)\rangle.$$

In the r.h.s. we have

$$\begin{aligned} (\widehat{L}(\mu) - \widehat{L}(\lambda))|E^R(\mu)\rangle \circ \left(\tilde{I} - \widetilde{R}\right) &= -\hat{\sigma} \int_{-\infty}^{\infty} |E^R(\mu)\rangle\langle F^L(\mu)|F^R(\lambda)\rangle \, d\mu \\ &= -\hat{\sigma}\cdot\widehat{B}|F^R(\lambda)\rangle. \end{aligned}$$

Therefore

$$\partial_x |F^R(\lambda)\rangle = \left(\widehat{L}(\lambda) + [\widehat{B}, \hat{\sigma}]\right)|F^R(\lambda)\rangle,$$

or

$$\partial_x |F^R(\lambda)\rangle = \widehat{\mathcal{L}}(\lambda)|F^R(\lambda)\rangle, \tag{5.5}$$

where

$$\widehat{\mathcal{L}}(\lambda) = \lambda\hat{\sigma} + [\hat{b}, \hat{\sigma}], \tag{5.6}$$

and

$$\hat{b} = \widehat{B} + \hat{g}. \tag{5.7}$$

In the same way one can obtain the similar formula for $\langle F^L(\lambda)|$

$$\partial_x \langle F^L(\lambda)| = -\langle F^L(\lambda)|\widehat{\mathcal{L}}(\lambda).$$

Now let us turn to the derivative of $|F^R(\lambda)\rangle$ with respect to t. As before, we start with differentiation of the kernel $V(\lambda, \mu)$ using formulæ (3.14)

$$\begin{aligned}
\partial_t V(\lambda, \mu) &= \frac{\langle E^L(\lambda)|(\lambda\widehat{L}(\lambda) - \mu\widehat{L}(\mu))|E^R(\mu)\rangle}{\lambda - \mu} \\
&= \frac{\langle E^L(\lambda)|((\lambda - \mu)\widehat{L}(\lambda) - \mu(\widehat{L}(\mu) - \widehat{L}(\lambda))|E^R(\mu)\rangle}{\lambda - \mu} \\
&= \langle E^L(\lambda)|\widehat{L}(\lambda)|E^R(\mu)\rangle + \mu\langle E^L(\lambda)|\hat{\sigma}|E^R(\mu)\rangle \\
&= -\partial_x\langle E^L(\lambda)| \cdot |E^R(\mu)\rangle + \mu\langle E^L(\lambda)|\hat{\sigma}|E^R(\mu)\rangle.
\end{aligned} \tag{5.8}$$

Thus, from (4.3) we get

$$\begin{aligned}
&\partial_t |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) - \int_{-\infty}^{\infty} |F^R(\lambda)\rangle \partial_x \langle E^L(\lambda)| \cdot |E^R(\mu)\rangle \, d\lambda \\
&+\mu \int_{-\infty}^{\infty} |F^R(\lambda)\rangle\langle E^L(\lambda)|\hat{\sigma}|E^R(\mu)\rangle \, d\lambda = [-\mu\widehat{L}(\mu) + \partial_x \hat{g}]|E^R(\mu)\rangle.
\end{aligned} \tag{5.9}$$

Comparing (5.9) with (5.4), we see that

$$\begin{aligned}
&\mu \int_{-\infty}^{\infty} |F^R(\lambda)\rangle\langle E^L(\lambda)|\hat{\sigma}|E^R(\mu)\rangle \, d\lambda + \mu\widehat{L}(\mu)|E^R(\mu)\rangle \\
&\qquad = \partial_x |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) \cdot \mu.
\end{aligned}$$

Substituting this formula into (5.9) we find

$$\partial_t |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) + \partial_x |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) \cdot \mu$$

$$= \int_{-\infty}^{\infty} |F^R(\lambda)\rangle \partial_x \langle E^L(\lambda)| \cdot |E^R(\mu)\rangle \, d\lambda + \partial_x \hat{g} |E^R(\mu)\rangle. \tag{5.10}$$

Acting on (5.10) by the resolvent from the right we have

$$\partial_t |F^R(\lambda)\rangle + \partial_x |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) \cdot \mu \circ \left(\tilde{I} - \widetilde{R}\right)$$

$$= \left(\int_{-\infty}^{\infty} |F^R(\mu)\rangle \partial_x \langle E^L(\mu)| \, d\mu \right) \cdot |F^R(\lambda)\rangle + \partial_x \hat{g} |F^R(\lambda)\rangle. \tag{5.11}$$

The second term in the l.h.s. of (5.11) is equal to

$$\partial_x |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) \cdot \mu \circ \left(\tilde{I} - \widetilde{R}\right)$$

$$= \partial_x |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) \cdot (\mu - \lambda + \lambda) \circ \left(\tilde{I} - \widetilde{R}\right)$$

$$= \lambda \partial_x |F^R(\lambda)\rangle - \partial_x |F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) \circ \langle F^L(\mu) | F^R(\lambda)\rangle$$

$$= \lambda \partial_x |F^R(\lambda)\rangle - \left(\int_{-\infty}^{\infty} \partial_x |F^R(\mu)\rangle \cdot \langle E^L(\mu)| \, d\mu \right) |F^R(\lambda)\rangle.$$

Therefore we arrive at

$$\begin{aligned} \partial_t |F^R(\lambda)\rangle \quad &= \quad -\lambda \partial_x |F^R(\lambda)\rangle + \left(\int_{-\infty}^{\infty} \partial_x |F^R(\mu)\rangle \cdot \langle E^L(\mu)| \, d\mu \right) \cdot |F^R(\lambda)\rangle \\ &+ \quad \left(\int_{-\infty}^{\infty} |F^R(\mu)\rangle \cdot \partial_x \langle E^L(\mu)| \, d\mu \right) \cdot |F^R(\lambda)\rangle + \partial_x \hat{g} |F^R(\lambda)\rangle \\ &= \quad -\lambda \partial_x |F^R(\lambda)\rangle + \partial_x (\widehat{B} + \hat{g}) |F^R(\lambda)\rangle, \end{aligned} \tag{5.12}$$

or

$$\partial_t |F^R(\lambda)\rangle = \hat{\mathcal{M}}(\lambda) |F^R(\lambda)\rangle. \tag{5.13}$$

Here

$$\hat{\mathcal{M}}(\lambda) = -\lambda \widehat{\mathcal{L}}(\lambda) + \partial_x \hat{b}, \tag{5.14}$$

or, using (5.6) we can rewrite (5.14) as

$$\hat{\mathcal{M}}(\lambda) = -\lambda^2 \hat{\sigma} - \lambda [\hat{b}, \hat{\sigma}] + \partial_x \hat{b}. \tag{5.15}$$

In the same way one can obtain the similar formula for $\langle F^L(\lambda)|$

$$\partial_t \langle F^L(\lambda)| = -\langle F^L(\lambda)|\hat{\mathcal{M}}(\lambda).$$

Thus, we have constructed the Lax representation.

6 The logarithmic derivatives of the determinant and the differential equations

Remind the reader that the operator describing the correlation function contains the projector $\widetilde{Y}$:

$$Y(\lambda,\mu) = P(\lambda)P(\mu) = \int_{-\infty}^{\infty} dudv E_+(\lambda|v)E_+(\mu|u).$$

In the operator notations it can be written in the form

$$\int_{-\infty}^{\infty} dudv E_+(\lambda|v)E_+(\mu|u) = \langle E^L(\lambda)|\hat{\sigma}_-|E^R(\mu)\rangle, \tag{6.1}$$

where the operator $\hat{\sigma}_-$ is equal to

$$\hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(This is the particular case of operator — its discrete components are constant functions of continuous indices u and v. The action of this operator on vectors is still given by (3.12)).

Let us calculate the derivative of $\det\left(\tilde{I} + \widetilde{V} - \frac{\alpha}{2\pi}\widetilde{Y}\right)$ with respect to α

$$\partial_\alpha \log\det\left(\widetilde{I} + \widetilde{V} - \frac{\alpha}{2\pi}\widetilde{Y}\right)\Big|_{\alpha=0}$$

$$= -\frac{1}{2\pi}\,\mathrm{Tr}\left(\langle E^L(\lambda)|\hat{\sigma}_-|E^R(\mu)\rangle \circ \left(\tilde{I} - \widetilde{R}\right)\right) = -\frac{1}{2\pi}\,\mathrm{Tr}\left(\langle E^L(\lambda)|\hat{\sigma}_-|F^R(\lambda)\rangle\right)$$

$$= -\frac{1}{2\pi}\,\mathrm{tr}\left(\widehat{B}\hat{\sigma}_-\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} B_{12}(u,v)\,dudv.$$

Here we denote by the symbol "Tr" the trace of the integral kernels. Hence,

$$\partial_\alpha \det\left(\widetilde{I} + \widetilde{V} - \frac{\alpha}{2\pi}\widetilde{Y}\right)\Big|_{\alpha=0} = -\det\left(\tilde{I} + \widetilde{V}\right)\int_{-\infty}^{\infty} \frac{dudv}{2\pi} B_{12}(u,v). \tag{6.2}$$

Due to the definitions (3.18), (5.7) and identity

$$G(x,t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} dudv g(u,v), \tag{6.3}$$

we come to the equality

$$G(x,t) - \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} B_{12}(u,v)\, dudv = -\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} b_{12}(u,v)\, dudv.$$

Hence the correlation function is equal to

$$\langle \psi(0,0)\psi^\dagger(x,t)\rangle_T = -\frac{e^{-iht}}{2\pi}(0|\frac{\det\Big(\tilde{I}+\widetilde{V}\Big)}{\det(\hat{I}-\frac{1}{2\pi}\widetilde{K}_T)} \int\limits_{-\infty}^{\infty} b_{12}(u,v)\, dudv|0). \tag{6.4}$$

The logarithmic derivatives of the determinant of the operator $\Big(\tilde{I}+\widetilde{V}\Big)$ with respect to x and t are equal

$$\partial_x \log\det\Big(\tilde{I}+\widetilde{V}\Big) = \mathrm{Tr}\left(\partial_x\widetilde{V}\circ\Big(\tilde{I}-\widetilde{R}\Big)\right),$$

$$\partial_t \log\det\Big(\tilde{I}+\widetilde{V}\Big) = \mathrm{Tr}\left(\partial_t\widetilde{V}\circ\Big(\tilde{I}-\widetilde{R}\Big)\right).$$

Using (5.3) we have

$$\mathrm{Tr}\left(\partial_x\widetilde{V}\circ\Big(\tilde{I}-\widetilde{R}\Big)\right) = -\,\mathrm{Tr}\left(\langle E^L(\lambda)|\hat{\sigma}|E^R(\mu)\rangle\circ\Big(\tilde{I}-\widetilde{R}\Big)\right)$$

$$= -\int\limits_{-\infty}^{\infty} d\lambda(\langle E^L(\lambda)|\hat{\sigma}|F^R(\lambda)\rangle) = -\,\mathrm{tr}\Big(\widehat{B}\cdot\hat{\sigma}\Big),$$

and hence

$$\partial_x \log\det\Big(\tilde{I}+\widetilde{V}\Big) = -\,\mathrm{tr}\Big(\widehat{B}\cdot\hat{\sigma}\Big). \tag{6.5}$$

In the same method one can compute the time derivative

$$\mathrm{Tr}\left(\partial_t\widetilde{V}\circ\Big(\tilde{I}-\widetilde{R}\Big)\right)$$

$$= \mathrm{Tr}\left(\langle E^L(\lambda)|(\hat{\sigma}(2\lambda+\mu-\lambda)+[\hat{g},\hat{\sigma}])|E^R(\mu)\rangle\circ\Big(\tilde{I}-\widetilde{R}\Big)\right)$$

$$= \int\limits_{-\infty}^{\infty} d\lambda\Big(2\lambda\langle E^L(\lambda)|\hat{\sigma}|F^R(\lambda)\rangle + \langle E^L(\lambda)|[\hat{g},\hat{\sigma}]|F^R(\lambda)\rangle\Big)$$

$$-\int\limits_{-\infty}^{\infty} d\lambda d\mu\Big(\langle E^L(\lambda)|\hat{\sigma}|E^R(\mu)\rangle\langle F^L(\mu)|F^R(\lambda)\rangle\Big)$$

$$= \mathrm{tr}\Big(2\widehat{C}\cdot\hat{\sigma} - \widehat{B}^2\cdot\hat{\sigma} + \widehat{B}\cdot[\hat{g},\hat{\sigma}]\Big) = \mathrm{tr}\Big(2(\widehat{C}+\hat{b}\cdot\hat{g})\cdot\hat{\sigma} - \hat{b}^2\cdot\hat{\sigma}\Big).$$

Therefore

$$\partial_t \log\det\Big(\tilde{I}+\widetilde{V}\Big) = \operatorname{tr}\Big(2(\widehat{C}+\hat{b}\cdot\hat{g})\cdot\hat{\sigma}-\hat{b}^2\cdot\hat{\sigma}\Big). \tag{6.6}$$

The second logarithmic derivatives of the determinant can be expressed in terms of matrix $\hat{b}$ only. Indeed, using (A.5) and (A.7) we get

$$\partial_x\partial_x \log\det\Big(\tilde{I}+\widetilde{V}\Big) = -\operatorname{tr}\Big([\hat{b},\hat{\sigma}]\cdot\hat{b}\cdot\hat{\sigma}\Big). \tag{6.7}$$

$$\partial_x\partial_t \log\det\Big(\tilde{I}+\widetilde{V}\Big) = -\operatorname{tr}\Big(\partial_x\hat{b}\cdot\hat{b}\cdot\hat{\sigma}-\hat{b}\cdot\partial_x\hat{b}\cdot\hat{\sigma}\Big). \tag{6.8}$$

Thus, we have expressed the logarithmic derivatives of the Fredholm determinant in terms of the traces of operators $\hat{b}$, $\widehat{B}$ and $\widehat{C}$. It is worth mentioning that the second logarithmic derivatives depend on the operator $\hat{b}$ only.

Now let us turn back to the Lax representation. The relations (5.5) and (5.13)

$$\partial_x|F^R(\lambda)\rangle = \widehat{\mathcal{L}}(\lambda)|F^R(\lambda)\rangle,$$

$$\partial_t|F^R(\lambda)\rangle = \hat{\mathcal{M}}(\lambda)|F^R(\lambda)\rangle$$

should be compatible. It means that

$$\partial_t\widehat{\mathcal{L}}-\partial_x\hat{\mathcal{M}}+[\widehat{\mathcal{L}},\hat{\mathcal{M}}]=0.$$

Substituting (5.6) and (5.15) into the last formula, we get

$$[\partial_t\hat{b},\hat{\sigma}]-\partial_x\partial_x\hat{b}+\Big[[\hat{b},\hat{\sigma}],\partial_x\hat{b}\Big]=0. \tag{6.9}$$

Remark. More accurately the equation (6.9) is valid if there exists a sequence of complex numbers λ_n such that the corresponding sequence of vectors $|F^R(\lambda_n)\rangle$ generates a basis of the space $\mathcal{H}$. However, one can check that this condition is not necessary. Indeed, using identities for potentials from Appendix A (in particular formulæ (A.3) and (A.6)) it is possible to prove the equality (6.9) directly, without using the compatibility condition.

Thus, we have arrived at the following results. The second logarithmic derivatives of the Fredholm determinant are presented in terms of the operator $\hat{b}$. On the other hand this operator satisfies the partial differential equation (6.9). In turn this equation appears to be the compatibility condition of the auxiliary linear problem (5.5), (5.13).

7 Main results in components of operators $\hat{b}$ and $\widehat{C}$

In this Section we summarize the main results obtained in the previous Sections. Till now we did not use the symmetry of the kernel $V(\lambda,\mu) = V(\mu,\lambda)$ (3.3). Using this property and the definitions (3.9) one can get additional identity

$$B_{11}(u,v) = -B_{22}(v,u). \tag{7.1}$$

Due to the definitions (5.7) and (3.18) we have

$$\begin{aligned} b_{ab}(u,v) &= B_{ab}(u,v), \qquad \text{for all } a,b \text{ exept the case } a=1,b=2, \\ b_{12}(u,v) &= B_{12}(u,v) - g(u,v) \end{aligned} \tag{7.2}$$

It follows from (A.3), (A.4) and the definition (3.17) that

$$\partial_x B_{11}(u,v) = i\int\limits_{-\infty}^{\infty} dw b_{12}(u,w) b_{21}(w,v), \tag{7.3}$$

$$\partial_x B_{22}(u,v) = -i\int\limits_{-\infty}^{\infty} dw b_{21}(u,w) b_{12}(w,v). \tag{7.4}$$

Using identities (A.5), (A.7) and expressions for the logarithmic derivatives of the determinant (6.5)–(6.8) we have

$$\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) = i\int\limits_{-\infty}^{\infty} du b_{11}(u,u), \tag{7.5}$$

$$\begin{aligned} \partial_t \log\det\left(\tilde{I}+\widetilde{V}\right) &= i\int\limits_{-\infty}^{\infty} dudv\Big(-b_{21}(u,v)g(v,u) \\ &+ i\int\limits_{-\infty}^{\infty} du(C_{22}(u,u) - C_{11}(u,u))\Big), \end{aligned} \tag{7.6}$$

$$\partial_x\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) = -\int\limits_{-\infty}^{\infty} dudv b_{12}(u,v) b_{21}(v,u), \tag{7.7}$$

$$\begin{aligned} \partial_t\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) &= i\int\limits_{-\infty}^{\infty} dudv(\partial_x b_{12}(u,v)\cdot b_{21}(v,u) \\ &- \partial_x b_{21}(u,v)\cdot b_{12}(v,u)). \end{aligned} \tag{7.8}$$

The diagonal part of the compatibility condition (6.9) is equal to zero due to (7.3) and (7.4). For the off-diagonal part of the compatibility condition we find

$$\begin{aligned} -i\partial_t b_{12}(u,v) &= -\partial_x^2 b_{12}(u,v) \\ &+2\int_{-\infty}^{\infty} dw_1 dw_2 b_{12}(u,w_1)b_{21}(w_1,w_2)b_{12}(w_2,v), \end{aligned} \tag{7.9}$$

$$\begin{aligned} i\partial_t b_{21}(u,v) &= -\partial_x^2 b_{21}(u,v) \\ &+2\int_{-\infty}^{\infty} dw_1 dw_2 b_{21}(u,w_1)b_{12}(w_1,w_2)b_{21}(w_2,v). \end{aligned} \tag{7.10}$$

The system of partial differential equations (7.9) and (7.10) is continuum generalization of the classical nonlinear Schrödinger equation. The second logarithmic derivatives of the Fredholm determinant (7.7) and (7.8) give us the densities of the first and the second integrals of motion of this equation. It means that the correlation function possess the properties of a τ-function of the generalized nonlinear Schrödinger equation.

Summary

The main goal of this paper was a description of the quantum correlation function in terms of solutions of the classical completely integrable equation. We have constructed such differential equations—(7.9), (7.10), which, in fact, are continual generalization of nonlinear Schrödinger equation. The correlation function of local fields plays role of the τ-function of these equations. In particular the second logarithmic derivatives of the Fredholm determinant (7.7) and (7.8) give us the densities of the first and the second integrals of motion. In a forthcoming publication we shall formulate the Riemann–Hilbert problem for the differential equation obtained in this paper. This permit us to evaluate the long distance asymptotic.

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A Identities for potentials

Here we establish some identities between matrix elements of potentials $\widehat{B}$ (or $\hat{b}$) and $\widehat{C}$. We also introduce the operator $\widehat{D}$:

$$\widehat{D} = \int\limits_{-\infty}^{\infty} \lambda^2 |F^R(\lambda)\rangle\langle E^L(\lambda)|\, d\lambda. \tag{A.1}$$

Let us calculate the derivative of the operator $\widehat{B}$ with respect to x. We shall use formulæ (3.13), (3.15), (5.5) and (5.6). We have

$$\partial_x \widehat{B} = \int\limits_{-\infty}^{\infty} d\lambda \left(\widehat{\mathcal{L}}(\lambda)|F^R(\lambda)\rangle\langle E^L(\lambda)| - |F^R(\lambda)\rangle\langle E^L(\lambda)|\widehat{L}(\lambda) \right)$$

$$= \int\limits_{-\infty}^{\infty} d\lambda \left((\lambda\hat{\sigma} + [\hat{b}, \hat{\sigma}])|F^R(\lambda)\rangle\langle E^L(\lambda)| - |F^R(\lambda)\rangle\langle E^L(\lambda)|(\lambda\hat{\sigma} + [\hat{g}, \hat{\sigma}]] \right)$$

$$= [\hat{\sigma}, \widehat{C}] + [\hat{b}, \hat{\sigma}] \cdot \widehat{B} - \widehat{B} \cdot [\hat{g}, \hat{\sigma}]. \tag{A.2}$$

Therefore

$$\partial_x \widehat{B} = [\hat{\sigma}, \widehat{C}] + [\hat{b}, \hat{\sigma}] \cdot \widehat{B} - \widehat{B} \cdot [\hat{g}, \hat{\sigma}]. \tag{A.3}$$

Taking the trace of this equality we have

$$\mathrm{tr}\Big(\partial_x \widehat{B}\Big) = 0. \tag{A.4}$$

Here we used definition $\hat{b} = \widehat{B} + \hat{g}$. Multiplying (A.3) by $\hat{\sigma}$ and taking the trace we have

$$\mathrm{tr}\Big(\partial_x \widehat{B} \cdot \hat{\sigma}\Big) = \mathrm{tr}\Big([\hat{b}, \hat{\sigma}] \cdot \hat{b} \cdot \hat{\sigma}\Big). \tag{A.5}$$

Here we used the property

$$\hat{g}^2 = (\hat{g}\hat{\sigma})^2 = 0, \qquad \text{see (3.18)}.$$

Using the same method one can calculate the derivative of the operator $\widehat{B}$ with respect to t. To do it we need formulæ (3.14), (3.16), (5.13) and (5.14). We have

$$\partial_t \widehat{B} = \int\limits_{-\infty}^{\infty} d\lambda \left(\hat{\mathcal{M}}(\lambda)|F^R(\lambda)\rangle\langle E^L(\lambda)| - |F^R(\lambda)\rangle\langle E^L(\lambda)|\hat{M}(\lambda) \right)$$

$$= -[\hat{\sigma}, \widehat{D}] - [\hat{b}, \hat{\sigma}] \cdot \widehat{C} + \widehat{C} \cdot [\hat{g}, \hat{\sigma}] + \partial_x \hat{b} \cdot \widehat{B} - \widehat{B} \cdot \partial_x \hat{g}. \tag{A.6}$$

Again multiplying by $\hat{\sigma}$ and taking the trace we have

$$\mathrm{tr}\Big(\partial_t \widehat{B}\cdot\hat{\sigma}\Big) = \mathrm{tr}\Big([\hat{\sigma},\widehat{C}]\cdot\hat{g}\cdot\hat{\sigma} - [\hat{\sigma},\widehat{C}]\cdot\hat{\sigma}\cdot\hat{b} + \partial_x\hat{b}\cdot\widehat{B}\cdot\hat{\sigma} - \widehat{B}\cdot\partial_x\hat{g}\cdot\hat{\sigma}\Big).$$

(Here we used cyclic permutation under the sign "tr"). Substituting $[\hat{\sigma},\widehat{C}]$ from (A.3) we get after simple algebra

$$\mathrm{tr}\Big(\partial_t \widehat{B}\cdot\hat{\sigma}\Big) = \mathrm{tr}\Big(\partial_x\hat{b}\cdot\hat{b}\cdot\hat{\sigma} - \hat{b}\cdot\partial_x\hat{b}\cdot\hat{\sigma}\Big). \tag{A.7}$$

These identities were used for calculation of the second logarithmic derivatives of the Fredholm determinant in Section 6. One can also use the method described above in order to prove directly the equation (6.9)

$$[\partial_t\hat{b},\hat{\sigma}] - \partial_x\partial_x\hat{b} + \Big[[\hat{b},\hat{\sigma}],\partial_x\hat{b}\Big] = 0.$$

B Quantum equation as continual differential equation

Consider the quantum nonlinear Schrödinger equation

$$i\partial_t\psi = -\partial_x^2\psi + 2c\psi^\dagger\psi\psi, \tag{B.1}$$

where $\psi(x,t)$ is annihilation operator in the bosonic Fock space. Let us take a matrix element of this equation between the ground state at finite density $|\Omega\rangle$ and state with one hole $\langle h_0|$. We shall have in mind that the momentum of the hole k_0 is fixed.

$$i\partial_t\langle h_0|\psi|\Omega\rangle = -\partial_x^2\langle h_0|\psi|\Omega\rangle + 2c\langle h_0|\psi^\dagger|h_0,h_1\rangle\langle h_1,h_0|\psi|h_2\rangle\langle h_2|\psi|\Omega\rangle. \tag{B.2}$$

Here h_2 is a hole with momentum k_2 and h_1 is a hole with momentum k_1. In the last term in the r.h.s. of (B.2) we have to integrate with respect to k_1 and k_2. This shows that quantum nonlinear Schrödinger equation is equivalent to the classical continual differential equation. The same type of equations describes quantum correlation functions (see (7.9), (7.10)).

C Free fermion limit

In the free fermion limit $c\to\infty$ function $E_-(\lambda|u)$ (3.2) does not depend on continuous variable u. Indeed, in this limit we have $h(\lambda,\mu) = (\lambda-\mu+ic)/ic \to 1$, hence all commutators (2.4) are equal to zero. Thus, one can put all dual fields equal to zero in expressions for functions $E_\pm$

$$E_+(\lambda|u) = \frac{\sqrt{\theta(\lambda)}}{2\pi}\left(\frac{1}{u-\lambda+i0} + \frac{1}{u-\lambda-i0}\right)e^{\tau(u)-\frac{1}{2}\tau(\lambda)}, \tag{C.1}$$

$$E_-(\lambda|u) = \frac{1}{\pi}e^{-\frac{1}{2}\tau(\lambda)}\sqrt{\theta(\lambda)}. \tag{C.2}$$

Hence matrix elements of the operator $\widehat{B}$ possess the following properties: $B_{21}(u,v)$ does not depend on u and v; $B_{11}(u,v)$ depends only on the first argument u; $B_{22}(u,v)$ depends only on the second argument v; $B_{12}(u,v)$ depends on both arguments u and v. Analogous properties have matrix elements of the operator $\widehat{C}$.

One can make the replacement

$$\begin{array}{ll}
B_{21}(u,v) \to B_{21}, & C_{21}(u,v) \to C_{21}, \\
\int\limits_{-\infty}^{\infty} du B_{11}(u,v) \to B_{11}, & \int\limits_{-\infty}^{\infty} du C_{11}(u,v) \to C_{11}, \\
\int\limits_{-\infty}^{\infty} dv B_{22}(u,v) \to B_{22}, & \int\limits_{-\infty}^{\infty} dv C_{22}(u,v) \to C_{22}, \\
\int\limits_{-\infty}^{\infty} du dv B_{12}(u,v) \to B_{12}, & \int\limits_{-\infty}^{\infty} du dv C_{12}(u,v) \to C_{12}.
\end{array}$$

Here functions B_{ab} and C_{ab} are scalar potentials. In terms of these potentials one can rewrite the correlation function, the logarithmic derivatives of determinant and the partial differential equations in the form

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T = -\frac{e^{-iht}}{2\pi} b_{12} \det\left(\tilde{I}+\widetilde{V}\right).$$

The logarithmic derivatives of the determinant

$$\begin{aligned}
\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) &= i b_{11}, \\
\partial_t \log\det\left(\tilde{I}+\widetilde{V}\right) &= i(-C_{11}+C_{22}-b_{21}G), \\
\partial_x\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) &= -b_{12}b_{21}, \\
\partial_t\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) &= i(\partial_x b_{12}\cdot b_{21} - \partial_x b_{21}\cdot b_{12}).
\end{aligned}$$

The differential equations.

(One should integrate (7.9) with respect to u and v)

$$\begin{aligned}
-i\partial_t b_{12} &= -\partial_x^2 b_{12} + 2b_{12}^2 b_{21}, \\
i\partial_t b_{21} &= -\partial_x^2 b_{21} + 2b_{21}^2 b_{12}.
\end{aligned}$$

Here $b_{ab} = B_{ab}$ for all a and b except $a=1, \quad b=2$; $b_{12} = B_{12} - G$. These results reproduce the results of [6] up to notations and rescaling the distance x and the time t.

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